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Differences Among Three Deformed Oscillators

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The differences among quon operators, q_a-math oscillator operators and q-deformed oscillator operators are pointed out. The q-deformed ocsillator and q_a-math oscillator are constructed in terms of $q_q = 0$ quon.

KEY WORDS: quon; q-deformed oscillator; q_a-math oscillator.

1. INTRODUCTION

As we know, there are three special kinds of deformed algebras in quantum physics: they are respectively the algebras of q_a-math oscillator (Arik and Coon, 1976), q-deformed oscillator (Biedenharn, 1989; Macfarlane, 1989; Sun and Fu, 1989) and quon (Greenberg, 1990a,b, 1991). These algebras are very useful in quantum group and intermediate statistics. Their commutations are similar, but they are different algebras. The q_a-math oscillator and q-deformed oscillator have been constructed by using ordinary oscillator as building block (Solomon, 1994), and the q-deformed oscillator was constructed in terms of $q_q = 0$ quon operators in Wu and Sun (1999).

In this paper, q_a-math oscillator, quon, and q-deformed oscillator are regarded as deformations of the usual harmonic oscillator and degenerate to it in three different limits. In fact, there are essential differences among q_a-math oscillator, q-deformed oscillator, and quon. Here we will reveal the relationships among q_a-math oscillator, quon, and q-deformed oscillator, and we find out that their parameter values have different domains. We shall construct q_a-math oscillator operators in terms of $q_q = 0$ quon operators.

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2. HILBERT SPACE STRUCTURE

2.1. q_a-Math Oscillator

Arik and Coon (1976) defined a special deformed mathematical oscillator and by introducing the operators $\{a, a^\dagger, N\}$ satisfied q_q-complete relation. In that paper, they only studied the single-mode case; here we generalize it to a multimode situation as follows.

The q_a-math oscillator is described by a series of operators $\{a_i, a_i^\dagger, {}_aN_i \mid i = 1, 2, \ldots\}$ in Hilbert space $\mathcal{A}$, subject to the following relations

$$\begin{aligned} a_i a_i^\dagger - q_a a_i^\dagger a_i &= 1, \\ [a_i, a_i^\dagger] &= 0 \ (i \neq j), \\ [a_i, a_j] = [a_i^\dagger, a_j^\dagger] &= 0, \\ [{}_aN_i \ a_j] &= -a_i \delta_{ij}, \\ [{}_aN_i \ a_j^\dagger] &= a_i^\dagger \delta_{ij}, \end{aligned} \tag{1}$$

where the deformation parameter $q_a \in (0, \infty)$, as in Arik and Coon (1976).

Since the different modes are independent, $\mathcal{A}$ is a symmetric space. An orthonormal basis of $\mathcal{A}$ can be chosen as

$$\begin{aligned} {}_aN_i |m_1, m_2, \ldots, m_n\rangle &= m_i |m_1, m_2, \ldots, m_n\rangle, \\ |m_1, m_2, \ldots, m_n\rangle &= \left(\prod_{i=1}^{n} \frac{a_i^{\dagger m_i}}{\sqrt{\{m_i\}!}} \right) |\ \rangle, \quad (n \geq 0), \end{aligned} \tag{2}$$

with

$$\{x\} = \frac{1 - q_a^x}{1 - q_a}; \quad \{n\}! \equiv \{n\}\{n-1\} \cdots \{1\}, \quad \{0\}! = 1. \tag{3}$$

As we know that any Hilbert space of physics must be positive definite, so we can calculate the domain of the deformed parameter as

$$q_a \in [-1, \infty]. \tag{4}$$

2.2. q-Deformed Oscillator

The q-deformed oscillator (Biedenharn, 1989; Macfarlane, 1989; Sun and Fu, 1989) is an associative algebra generated by $\{b_i, b_j^\dagger, N_i \mid i = 1, 2, 3, \ldots\}$ satisfying

$$\begin{aligned} b_i b_i^\dagger - q b_i^\dagger b_i = q^{N_i}, &\qquad [b_i, b_j^\dagger] = 0; (i \neq j), \\ [N_i, b_j] = -\delta_{ij} b_i, &\qquad [N_i, b_j^\dagger] = \delta_{ij} a_i^\dagger. \end{aligned} \tag{5}$$

From the q-deformed oscillator's commutation, we see that Hilbert space $\mathcal{B}$ is a symmetric space. So an orthonormal basis of $\mathcal{B}$ can be chosen as

$$|m_1, m_2, \ldots, m_n\rangle = \left(\prod_{i=1}^{n} \frac{b_i^{\dagger m_i}}{\sqrt{[m_i]!}}\right) |\,\rangle, \quad (n \geq 0), \tag{6}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}; \quad [n]! \equiv [n][n-1]\cdots[1], \quad [0]! = 1. \tag{7}$$

Hilbert space $\mathcal{B}$ is positive definite too. So the domain of the parameter q is given by

$$\begin{aligned} &\text{Case 1}: q > 0, \\ &\text{Case 2}: q = e^{i\theta} \ \text{ where } \ 0 \leq \theta \leq \pi/\dim \mathcal{B}. \end{aligned} \tag{8}$$

One can verify that to Case 1 the dimension of the Hilbert space is infinite, and to Case 2, the dimension is finite.

Both the Hilbert space $\mathcal{A}$ and $\mathcal{B}$ are symmetric, so we easily find the relationship between q_a-math oscillator and q-deformed oscillator, as follows

$$b_i = \sqrt{\frac{[{}_aN_i + 1]}{\{{}_aN_i + 1\}}}a_i \qquad b_i^\dagger = a^\dagger \sqrt{\frac{[{}_aN_i + 1]}{\{{}_aN_i + 1\}}} \tag{9}$$

So we get the relation between q_a-math oscillator and q-deformed oscillator.

2.3. Quon Algebra

Quon (Grenberg, 1990a,b, 1991) is described by a series of operators $\{c_i, c_i^\dagger, {}_qN_i \mid i = 1, 2, \ldots\}$ in Hilbert space $\mathcal{G}$, subject to the following relations

$$c_i c_j^\dagger - q_q c_j^\dagger c_i = \delta_{ij}. \tag{10}$$

For the single-mode case, this is the same as the mathematical deformation of the harmonic oscillator introduced by Arik and Coon (1976). However, for the multimode quon, we note that different modes are not independent, namely $c_i c_j^\dagger = q_q c_j^\dagger c_i$ for $i \neq j$. Define a normalized vacuum state $|\rangle$ in Hilbert space $\mathcal{G}$, which is annihilated by all quon annihilation operators $c_i\ |\rangle = 0\,(i = 0, 1, 2, \ldots)$. Then the vacuum state $|\rangle$ and all the states $c_{j_1}^\dagger c_{j_2}^\dagger \cdots c_{j_m}^\dagger\ |\rangle\,(m > 1)$ span the whole Hilbert space $\mathcal{G}$. As a Hilbert space, $\mathcal{G}$ should be positive and this requirement gives a limitation of the parameter q_q: $q_q \in [-1, \infty]$ for single-mode quon, and $q_q \in [-1, 1]$ for the multimode case (Yu and Wu, 1994).

For the multimode and $q_q = 0$ case, we have

$$c_i c_j^\dagger = \delta_{ij} \tag{11}$$

and

$$\begin{aligned} c_i|\alpha_1\alpha_2\cdots\alpha_m\rangle &= \delta_{i1}|\alpha_2\cdots\alpha_m\rangle, \\ c_j^\dagger|\alpha_1\cdots\alpha_m\rangle &= |\alpha_i\alpha_1\cdots\alpha_m\rangle. \end{aligned} \tag{12}$$

In this case, we do not have relations $[c_i, c_j] = 0$ and $[c_i^\dagger, c_j^\dagger] = 0$ when $i \neq j$. This means the Hilbert space $\mathcal{G}$ is not a symmetric space. In fact, taking two arbitrary states $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$ in $\mathcal{G}$, we can easily find from Eqs. (12) that $\langle\alpha\beta \mid \beta\alpha\rangle = \delta_{\alpha\beta}$; in other words, $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$ are linearly independent.

3. q_a-MATH OSCILLATOR IN TERMS OF $q_q = 0$ QUON

From Section 2 we know q_a-math oscillator, q-deformed oscillator, and quon have completely different Hilbert space structures. However, we can restrict ourself only in the symmetric subspace of $\mathcal{G}$, which is respectively isomorphic to the spaces $\mathcal{A}$ and $\mathcal{B}$, and one can express q_a-math oscillator and q-deformed oscillator operators in terms of quon operators with $q_q = 0$. We first consider the simple single-mode case.

3.1. Single-Mode Case

The Hilbert space $\mathcal{G}$ becomes a symmetrical space for the single-mode case. Define an operator ${}_qN_0$ of quon in the Hilbert space $\mathcal{G}$ by

$${}_qN_0 = \sum_{k=1}^{\infty}(c^\dagger)^k c^k, \tag{13}$$

which counts the number of quons

$${}_qN_0|n\rangle_0 = n|n\rangle_0.$$

All the Hilbert spaces of single-mode q_a-math oscillator, single-mode q-deformed oscillator, and single-mode quon with $q_q = 0$ are symmetric spaces, their bases are $|m\rangle = \frac{(a^\dagger)^m}{\sqrt{\{m\}!}}|\rangle$ and $|n\rangle_0 = (c^\dagger)^n|\ \rangle$. From this and Eqs. (2) and (6) we can construct q_a-math oscillator and q-deformed oscillator operators by using quon with $q_q = 0$ operators as building blocks,

$$\begin{aligned} a &= \sqrt{\{{}_qN_0 + 1\}}c, \qquad a^\dagger = c^\dagger\sqrt{\{{}_qN_0 + 1\}}, \\ b &= \sqrt{[{}_qN_0 + 1]}c, \qquad b^\dagger = c^\dagger\sqrt{[{}_qN_0 + 1]}. \end{aligned} \tag{14}$$

3.2. Multimode Case

At first, from Greenberg (1990a,b, 1991), we know that multimode quon number operators ${}_qN$ and ${}_qN_i$ are

$$
\begin{aligned}
{}_qN_i &= c_i^\dagger c_i + \sum_{k=1}^{\infty} c_k^\dagger c_i^\dagger c_i c_k + \cdots + \sum_{k_1\cdots k_s}^{\infty} c_{k_1}^\dagger \cdots c_{k_s}^\dagger c_i^\dagger c_i c_{k_s} \cdots c_{k_1} + \cdots, \\
{}_qN &= \sum_{i=1}^{\infty} ({}_qN_i).
\end{aligned}
\tag{15}
$$

The states of M bosons are described as

$$
|n_1 \cdots n_m\rangle^s = \frac{1}{\sqrt{M!n_1!\cdots n_m!}} \sum_{p\in S_M} \left|\alpha_{p(1)} \cdots \alpha_{p(M)}\right\rangle, \tag{16}
$$

where S_M is the symmetric group of order M. While the annihilation operator of q-math oscillator in Dirac symbol is

$$
\begin{aligned}
a_i &= \sum_{n_1\cdots n_M} \sqrt{\{{}_qN_i + 1\}} |n_1 \cdots n_i \cdots n_M\rangle^{ss} \langle n_1 \cdots n_i + 1 \cdots n_M| \\
&= \left(\frac{\{{}_qN_i + 1\}}{({}_qN + 1)({}_qN_i + 1)} \right)^{1/2} \sum_{n_1\cdots n_M} \sum_{P\in S_M} \sum_{Q\in Q_{M+1}} \frac{1}{M!n_1!\cdots n_M!} \\
&\quad \times c_{P(1)}^\dagger \cdots c_{P(M)}^\dagger |\ \rangle\langle\ | c_{Q(1)} \cdots c_{Q(M+1)}.
\end{aligned}
\tag{17}
$$

Using the vacuum projector of quon $|\ \rangle\langle\ | = 1 - \sum_{i=1}^{\infty} c_i^\dagger c_i$, we easily get

$$
\begin{aligned}
a_i = \sqrt{\frac{\{{}_qN_i + 1\}}{({}_qN + 1)(qN_i + 1)}} \Bigg(& c_i + \sum_k c_k^\dagger c_i c_k + \cdots \\
& + \sum_{k_1\cdots k_s} c_{k_1}^\dagger \cdots c_{k_s}^\dagger c_i c_{k_s} \cdots c_{k_1} + \cdots \Bigg) \Bigg|_{H_s}
\end{aligned}
\tag{18}
$$

In the same way, we can construct q_a-math oscillator creation operators using $q_q = 0$ quon operators as

$$
\begin{aligned}
a_i^\dagger = & \left(c_i^\dagger + \sum_k c_k^\dagger c_i^\dagger c_k + \cdots + \sum_{k_1\cdots k_s} c_{k_1}^\dagger \cdots c_{k_s}^\dagger c_i^\dagger c_{k_s} \cdots c_{k_1} + \cdots \right) \\
& \times \sqrt{\frac{\{{}_qN_i + 1\}}{({}_qN + 1)(qN_i + 1)}} \Bigg|_{H_s}
\end{aligned}
\tag{19}
$$

The symbol $|_{H_s}$ means the formula only adapts on the symmetric subspace of the whole Hilbert space in Eqs. (18) and (19).

4. CONCLUSION

In conclusion, we have revealed that q_a-math oscillator operators, q-deformed oscillator operators, and quon operators have different Hilbert space structures and different domains of the deformation parameter, and that they obey different statistics, both q_a-math oscillator and q-deformed oscillator obeys Boson–Einstein statistics, while, the quon obey Boson–Einstein statistics when $q_q = 1$ and Fermi–Dirac statistics when $q_q = -1$, and Boltzmann–Maxwell statistics when $q_q \in (-1, 1)$. We also constructed q_a-math oscillator in terms of $q_q = 0$ quon.

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